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Conformal invariance and critical exponents of the Takhtajan–Babujian models

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Abstract. This paper is concerned with the critical properties of antiferromagnetic Takhtajan-Babujian models with spin $S = 1$, $\frac{3}{2}$ and 2. The leading eigenenergies of this Hamiltonian, in a finite chain, are calculated by investigating numerically and analytically the Bethe ansatz equations for the finite system. The critical exponents and the conformal anomaly are obtained from their relations with the eigenspectrum of the finite Hamiltonian. The appearance of logarithmic corrections produces poor estimates. However, a combination of analytical and numerical methods produces very good estimates. Our results strongly support the conjecture that the Wess-Zumino-Witten-Novikov non-linear σ models with topological charge $k = 2S$ are the underlying field theories for these spin-S statistical mechanics models.

1. Introduction

Most of the statistical systems at criticality (Cardy 1987) are believed to satisfy the basic assumptions—short-range interactions, scale invariance, rotational and translational invariance—that ensure conformal invariance (Polyakov 1970). This symmetry powerfully constrains the possible universality classes for the critical behaviour of one-dimensional quantum systems (or equivalently finite-temperature two-dimensional classical systems) (Belavin *et al* 1984a, b, Friedan *et al* 1984). Within conformal theory these classes are characterised by a single dimensionless number c, the central charge or conformal anomaly of the associated Virasoro algebra, the irreducible representations of which determine the operator algebra describing the critical behaviour. If c is less than unity and the critical theory is unitary (Friedan *et al* 1984) both c and the critical exponents associated with the different correlations are quantised:

$$c = 1 - \frac{6}{m(m+1)} \qquad m = 3, 4, 5, \ldots. \tag{1.1}$$

The scaling dimensions $X = h + \bar{h}$ (related to the critical exponents) of the primary operators are given by the Kac formula

$$h_{p,q} = \frac{[p(m+1) - qm]^2 - 1}{4m(m+1)} \qquad 1 \leqslant q \leqslant p \leqslant m-1. \tag{1.2}$$

When $c = 1$ equations (1.1) hold with $m = \infty$, allowing $h_{p,q}$ to take any positive value. Indeed it appears that unitarity does not introduce any constraint on the possible values of c and h for $c \geqslant 1$. Nevertheless, when the primary operators obey a larger algebra than the Virasoro algebra a relationship between c and h may occur even for $c > 1$. Examples of these are the models satisfying supersymmetry (Friedan *et al* 1985, Bershadsky *et al* 1984), Zamolodchikov-Fateev algebra (Zamolodchikov and Fateev 1985) and Kac-Moody algebra (Knizhnik and Zamolodchikov 1984). In this last case we have the Wess-Zumino-Witten-Novikov model whose conformal anomaly is given in terms of the charge k and the particular semisimple group G which define the Kac-Moody algebra (Knizhnik and Zamolodchikov 1984)

$$c = \frac{kD}{C_v + k} \tag{1.3}$$

where D is the dimension of G, $C_v = \delta^{ab} f^{acd} f^{bcd}$ and f^{abc} are the structure constants of G. The scaling dimensions of the primary operators ($X = h + \bar{h}$) are given by

$$h = \frac{c_l}{C_v + K} \tag{1.4}$$

where $c_l = b_l^a t_l^a$ and t_l^a are the generators of the l representation of G.

On the other hand (Blöte *et al* 1986, Affleck 1986a) the low-temperature specific heat of an infinite quantum spin Hamiltonian is related to the central charge of the conformal algebra governing its critical behaviour. Using this fact Affleck (1986a, b) calculated the conformal anomaly for a set of integrable one-dimensional quantum Hamiltonians introduced simultaneously by Babujian (1982, 1983) and Takhtajan (1982):

$$c = 3S/(1+S). \tag{1.5}$$

These Takhtajan-Babujian models describe the dynamics of spin-S particles (see § 2) and their conformal anomaly (1.5) coincides with (1.3) when $k = 2S$, G = SU(2) ($D = 3$, $C_v = 2$). This fact and an approximate mapping between those models (Affleck 1986a, b, c, Affleck and Haldane 1987) lead to the conjecture that the Wess-Zumino-Witten-Novikov models with symmetry group SU(2) and topological charge $k = 2S$ are the underlying field theories describing the criticality of the spin-S Babujian-Takhtajan models. This implies from (1.4) (Affleck and Haldane 1987) that these spin models should have primary operators with scaling dimensions X_j given by

$$X_j = j(j+1)/(1+S) \qquad j = 0, \tfrac{1}{2}, 1, \ldots, S. \tag{1.6}$$

In this paper, in order to independently verify these conjectures, we will calculate directly the conformal anomaly and some of the scaling dimensions occurring in these spin models. These calculations will be done by exploiting a set of important relations between these quantities and the eigenspectrum of the Hamiltonian with a finite number L of spins. These relations are consequences (see Cardy (1987) for a recent review) of the conformal invariance of the infinite system at the critical point. The relevant relations, for our purposes, may be stated as follows. To each primary operator ϕ, with anomalous dimension X_ϕ and spin S_ϕ, in the operator algebra of the critical infinite chain, there exists a set of states in the quantum Hamiltonian, in a periodic chain of L sites, whose energy and momentum are given by

$$E_{n,n'} = E_0^{(0)} + \frac{2\pi\zeta}{L}(X_\phi + n + n') + o(L^{-1}) \qquad n, n' = 0, 1, 2, \ldots \tag{1.7a}$$

$$P_{n,n'} = \frac{2\pi}{L}(S_\phi + n - n') \qquad n, n' = 0, 1, 2, \ldots \tag{1.7b}$$

respectively as $L \to \infty$. The ground-state energy of the finite chain is denoted by $E_0^{(0)}$ and the constant (model-dependent) ζ is introduced in order to ensure that the resulting equations of motion are conformally invariant (von Gehlen *et al* 1986). In addition to the above relations, conformal invariance also predicts (Blöte *et al* 1986, Affleck 1986a) that, at criticality, the L-site ground-state energy $E_0^{(0)}$ should behave as

$$E_0^{(0)}/L = e_\infty - \tfrac{1}{6}\pi c\zeta/L^2 + o(L^{-2}) \tag{1.8}$$

as $L \to \infty$. Here c is the central charge of the conformal class governing the critical behaviour and e_∞ is the ground-state energy per particle in the bulk limit ($L \to \infty$).

Previous attempts to verify the above conjectures, for the spin-1 Takhtajan–Babujian model, based on finite-size calculations, were reported in the literature. The spin-1 model was studied for lattice sizes up to $L = 12$ (Blöte and Capel 1986, Bonner *et al* 1987, Blöte and Bonner 1987 and references therein). However, due to small system sizes and the presence of logarithmic corrections, as we shall see in § 3, these results produced no convincing numerical agreement with the above conjecture. The existence of a Bethe ansatz for these spin-S models (Babujian 1982, 1983, Takhtajan 1982) will allow us, in this paper, to calculate their eigenspectrum for much larger chain sizes. Analysing numerically and analytically their Bethe ansatz equations, in the case of spin $S = 1$, $\frac{3}{2}$ and 2, we were able to extract very good estimates for the conformal anomaly and several anomalous dimensions, rendering a very good test of the conjectures (1.5) and (1.6)†. An earlier calculation (Ziman and Schulz 1987), also based on a numerical analysis of the Bethe ansatz equations, was presented for the conformal anomaly and the first scaling dimension in (1.6) for the spin $S = \frac{3}{2}$ model, with good agreement with (1.5) and (1.6).

The layout of this paper is as follows. In § 2 we present the model as well as their Bethe ansatz (BA) equations. In § 3 we rederive these equations using the string assumption and show that they do not produce the correct finite-size corrections. In §§ 4 and 5 we discuss the numerical solutions of these BA equations and present our results. Finally our conclusion is presented in § 6 and the analytical calculation of the finite-size corrections of the energies, using the string hypothesis, is presented in an appendix.

2. The spin-S Takhtajan–Babujian model

Since the success of the Bethe ansatz in solving the spin-$S = \frac{1}{2}$ Heisenberg model (Bethe 1931) much effort has been expended in order to obtain generalisations of this model which preserve exact integrability through the Bethe ansatz (see, e.g., Lieb and Wu 1972, Thacker 1981, Baxter 1982, Gaudin 1983, Tsvelick and Weigmann 1983). A generalisation of the Heisenberg model to arbitrary spin S, preserving the SU(2) symmetry and integrability, was obtained by Takhtajan (1982) and Babujian (1982, 1983). These Takhtajan–Babujian models describe the dynamics of antiferromagnetic spins and for a L-site chain they are defined by the Hamiltonian

$$H_s = \sum_{n=1}^{L} Q_{2S}(\boldsymbol{S}_n \cdot \boldsymbol{S}_{n+1}) \tag{2.1a}$$

† A short account of our results in the case of spin 1 has already been presented (Alcaraz and Martins 1988a).

where $\boldsymbol{S}_n \equiv (S_n^x, S_n^y, S_n^z)$ are SU(2) operators of arbitrary integer or half-integer spin S attached at the site n. The function $Q_{2S}(x)$ is a special polynomial of degree $2S$ which ensures exact integrability and is defined by

$$Q_{2S}(x) = -J \sum_{l=0}^{2S} \sum_{k=l+1}^{2S} \left(\frac{1}{k}\right) \prod_{\substack{j=0 \\ j\neq l}}^{2S\,\prime} \left(\frac{x - x_j}{x_l - x_j}\right) \tag{2.1b}$$

where $x_l = \frac{1}{2}[l(l+1) - 2S(S+1)]$ and $J(>0)$ is the antiferromagnetic coupling constant. Apart from a harmless constant the case $S = \frac{1}{2}$ reduces to the well studied Heisenberg Hamiltonian

$$H_{1/2} = \tfrac{1}{4}J \sum_{n=1}^{L} \boldsymbol{S}_n \cdot \boldsymbol{S}_{n+1} \tag{2.2}$$

while in the cases $S = 1, \frac{3}{2}$ and 2, which we are interested in this paper, the Hamiltonians are

$$H_1 = \tfrac{1}{4}J \sum_{n=1}^{L} [\boldsymbol{S}_n \cdot \boldsymbol{S}_{n+1} - (\boldsymbol{S}_n \cdot \boldsymbol{S}_{n+1})^2] \tag{2.3}$$

$$H_{3/2} = J \sum_{n=1}^{L} [-\tfrac{3}{8} - \tfrac{1}{16}(\boldsymbol{S}_n \cdot \boldsymbol{S}_{n+1}) + \tfrac{1}{54}(\boldsymbol{S}_n \cdot \boldsymbol{S}_{n+1})^2 + \tfrac{1}{27}(\boldsymbol{S}_n \cdot \boldsymbol{S}_{n+1})^3] \tag{2.4}$$

and

$$H_2 = J \sum_{n=1}^{L} [-\tfrac{1}{4} + \tfrac{13}{48}(\boldsymbol{S}_n \cdot \boldsymbol{S}_{n+1}) + \tfrac{43}{864}(\boldsymbol{S}_n \cdot \boldsymbol{S}_{n+1})^2 - \tfrac{5}{432}(\boldsymbol{S}_n \cdot \boldsymbol{S}_{n+1})^3 - \tfrac{1}{288}(\boldsymbol{S}_n - \boldsymbol{S}_{n+1})^4] \tag{2.5}$$

respectively.

The general Hamiltonian (2.1), with periodic boundary conditions imposed, commutes with the total spin operator $\hat{S}^z = \Sigma_n S_n^z$ and consequently the associated Hilbert space can be separated, in the $\hat{S}^z$ basis, in $2LS+1$ disjoint sectors labelled by the eigenvalues of $\hat{S}^z$, namely $r = 0, \pm 1, \pm 2, \ldots, \pm LS$. We can restrict ourselves only to the sectors with $r \geq 0$ because the sectors $r = +k$ and $r = -k$ are degenerate due to the spin reversal symmetry of (2.1). In the Bethe ansatz formulation for these models (Takhtajan 1982, Babujian 1982) the eigenenergies, for a given sector r, will be given in terms of $(SL - r)$ complex roots $(\lambda_1, \lambda_2, \ldots, \lambda_{SL-r})$ of the non-linear set of Bethe ansatz equations

$$\left(\frac{\lambda_j - \mathrm{i}S}{\lambda_j + \mathrm{i}S}\right)^L = \prod_{k=1\neq j}^{SL-r} \left(\frac{\lambda_j - \lambda_k + i}{\lambda_j - \lambda_k + i}\right) \qquad j = 1, 2, \ldots, SL - r. \tag{2.6}$$

The energy and momentum of the eigenstates are

$$E = -J \sum_{j=1}^{SL-r} \frac{S}{\lambda_j^2 + S^2} \tag{2.7a}$$

and

$$P = \frac{1}{\mathrm{i}} \sum_{j=1}^{SL-r} \ln\left(\frac{\lambda_j - \mathrm{i}S}{\lambda_j + \mathrm{i}S}\right) \qquad (\mathrm{mod}\ 2\pi). \tag{2.7b}$$

The exact solution of the system (2.6), in the bulk limit (Takhtajan 1982, Babujian 1982, 1983), reveals that these models are gapless (critical) with an antiferromagnetic ground state with energy per site e_∞ (hereafter we assume $J=1$) given by

$$
\begin{aligned}
&e_\infty=-1 \qquad (S=1)\\
&e_\infty=-(\log 2+\tfrac{1}{2}) \qquad (S=\tfrac{3}{2})\\
&e_\infty=-\tfrac{4}{3} \qquad (S=2)
\end{aligned}
\tag{2.8}
$$

etc, and a momentum dispersion relation

$$\varepsilon_k=\tfrac{1}{2}\pi\sin(k) \qquad 0\leqslant k\leqslant\pi \tag{2.9}$$

independent of the spin S.

3. String hypothesis and the large L limit

A standard way of transforming the system of complex variables (2.6) into a set of real ones is the so-called string hypothesis, which asserts that as $L\to\infty$ the roots λ_j cluster in complex n-strings. Each n-string contains n complex roots of the form

$$\lambda_{j,k}^n=\lambda_j^n+\tfrac{1}{2}\mathrm{i}(n+1-2k) \qquad k=1,2,\ldots,n \tag{3.1}$$

where λ_j^n are real numbers corresponding to the centre of the n-strings. The above assumption allow us to parametrise an arbitrary configuration of roots by the numbers ν_n of strings of size n such that $\Sigma_n n\nu_n=SL-r$. The set of equations (2.6), for a given configuration of strings $\{\nu_m\}$, reduces to the system of equations for λ_j^n, $j=1,2,\ldots,\nu_n$ (see, for example, Babujian 1983)

$$L\psi_{n,2S}(\lambda_j^n)=2\pi Q_j^n+\sum_{m=1}^{\infty}\sum_{k=1}^{\nu_m}\Xi_{n,m}(\lambda_j^n-\lambda_k^m) \tag{3.2a}$$

where

$$\psi_{n,2S}(x)=\sum_{l=1}^{\min(m,2S)}\theta_{m+2S+1-2l}(x) \tag{3.2b}$$

$$\Xi_{n,m}(x)=\begin{cases}\theta_{|m-n|}(x)+2\theta_{|m-n|}(x)+\ldots+2\theta_{m+n-2}(x)+\theta_{m+n}(x) & m\neq n \qquad (3.2c)\\ 2\theta_2(x)+\ldots+2\theta_{2m-2}+\theta_{2m} & m=n \qquad (3.2d)\end{cases}$$

and

$$\theta_n(x)=2\tan^{-1}(x/n). \tag{3.2e}$$

The numbers Q_j^n are integers or half-integers, depending on the particular distribution $\{\nu_m\}$ of strings. The ground state, which occurs in the $r=0$ sector, corresponds to a sea of $2S$-strings ($\nu_{2S}=L/2$, $\nu_m=0$, $m\neq 2S$), while the lowest energy state in the sector $r=1$ corresponds to $\nu_{2S}=\frac{1}{2}L-1$, $\nu_{2S-1}=1$ and so on. The numbers Q_j^n that appear in (3.2) are integer or half-integers and arise because of the complex logarithms occurring in the deduction of (3.2) from (2.6) and (3.1). For example, for the lowest energy in the r sector ($r\leqslant 2S$), where $\nu_{2S}=\frac{1}{2}N-[r/2S]$, $\nu_{2S-\{r/2S\}}=1$ ($[r/2S]$ and $\{r/2S\}$ are the integer part and the rest of the ratio $r/2S$), these numbers are

$$Q_j^{2S}=-\tfrac{1}{2}(\nu_{2S}-1)+I-1 \qquad I=1,\ldots,\nu_{2S} \tag{3.3a}$$

$$Q_j^{2S-\{r/2S\}}=0. \tag{3.3b}$$

Within the string hypothesis, from (2.7a) and (3.1) the eigenenergies for a distribution $\{\nu_m\}$ of strings are given by

$$E_N = -\tfrac{1}{2} \sum_{k=1}^{\infty} \sum_{j=1}^{\nu_k} \psi'_{m,2S}(\lambda_j^k) \tag{3.4}$$

where, as usual, the prime indicates the derivative.

It is important to stress here that the assumption (3.1) is valid only in the $L \to \infty$ limit whenever the state contains n-strings ($n > 1$). As a consequence, we do not expect that the finite-size corrections to the eigenenergies, obtained using this assumption, will be correct. Although these corrections are not correct quantitatively they will give an idea of the true finite-size corrections of the eigenspectrum. These corrections, within the string hypothesis, can be calculated by using an analytical method developed by Woynarovich and Eckle (1987a). These calculations, although not trivial, are very lengthy and we explain their main steps in an appendix. The leading finite-size corrections are calculated for the lowest energy of the sector r $(0, 1, 2, \ldots)$. From (A23) the ground-state energy E_0^{st}, using the string hypothesis, for the L-site chain behaves, as $L \to \infty$ like

$$E_0^{\text{st}}/L = e_\infty - \frac{\pi^2}{12L^2} + \frac{a}{L^2(\ln L)^3} + \mathrm{O}\left(\frac{\ln(\ln L)}{L^2(\ln L)^4}\right) \tag{3.5a}$$

while from (A24) the mass gap corresponding to the lowest energy E_r^{st}, of sector r, behaves as $L \to \infty$ like

$$(E_r^{\text{st}} - E_0^{\text{st}})/L = \frac{\pi^2 r^2}{4SL^2} + \frac{b}{L^2 \ln L} + \mathrm{O}\left(\frac{\ln(\ln L)}{L^2(\ln L)^2}\right). \tag{3.5b}$$

The numbers a and b depend on the values of the spin S and sector r (Alcaraz and Martins 1988b).

On the other hand, using the value $\zeta = \pi/2$ (for all S), obtained from the dispersion relation (2.9) (von Gehlen *et al* 1986), in relations (1.7) and (1.8), with expressions (3.5) we obtain $c = 1$ for all values of the spin S and $X_r = r^2/4S$, $r = 1, 2, \ldots$, for the scaling dimensions of the operators occurring in the model. These results, as we already expected, are in complete contradiction with the conjectures (1.5) and (1.6) for $S > \frac{1}{2}$, because the string hypothesis in these cases is valid only in the infinite-size limit. In the case of $S = \frac{1}{2}$ equations (3.5) give us the correct results of $c = 1$ and $X_{1/2} = \frac{1}{2}$ because, in this case, E_r^{st} consists of a sea of particles (1-strings) and the string hypothesis is valid even for finite L. Although (3.5a, b) do not give us the correct results we expect that the logarithmic corrections presented in (3.5) will also be present in the true finite-size energies. These corrections will decrease the convergence rate of our estimates for the conformal anomaly and scaling dimensions, especially for the scaling dimensions where the corrections are stronger. These logarithmic corrections are exactly known in the Heisenberg model $S = \frac{1}{2}$ (Woynarovich and Eckle 1987a, b, Woynarovich 1987, Alcaraz *et al* 1987, 1988, Hamer *et al* 1987) and their presence in the model with arbitrary spin explains the poor estimates of previous finite-size calculations of the correlation function critical exponent η of the spin-1 model (2.3) (Blöte and Capel 1986, Bonner *et al* 1987, Blöte and Bonner 1987). As in the Heisenberg model we expect that these corrections occur because the operator governing the finite-size corrections is marginal for these spin models.

4. Numerical solution of the Bethe ansatz equations

In order to obtain the correct finite-size corrections of the lowest energies in the sectors $r = 0, 1, 2, \ldots,$ of the models (2.1) we solved numerically the original $(LS - r)$-coupled non-linear complex Bethe ansatz equations for spins $S = 1$, $\frac{3}{2}$ and 2. We solve these equations by using a Newton-type method. Firstly we solve the more simple set of real equations (3.2) and use this solution, together with (3.1), to produce an initial guess for the complex root system (2.6). In table 1 we exhibit the complex roots for the ground-state energy of the sixteen-site chain of spin $S = 2$. We also give in this table the string's centre coordinates, obtained by solving the corresponding equation (3.2). In tables 2 and 3 we also show these roots for the lowest state in the $r = 1$ sector of the spin-$\frac{3}{2}$ model with $L = 16$ and $L = 20$, respectively. In order to verify if the solutions we found numerically correspond to the lowest energy state, in a given sector r, we diagonalise the Hamiltonians (2.3)–(2.5) directly for small sizes ($L \sim 10$–12) and compare their energies with those obtained from (2.6).

On the other hand, we observe that, for certain values of the sector and spin, the string solution is not a very good initial guess in solving (2.6). For example, this occurs for the $r = 1$ sector of spin $\frac{3}{2}$. If we compare, for this case, the complex roots for $L = 16$

Table 1. Complex roots $\lambda_j = \lambda_j^R + i\lambda_j^I$ ($j = 1$–32) of the system (2.6) corresponding to the ground state of the spin $S = 2$ in a $L = 16$ site chain. The other roots not shown in the table are obtained from these by the combination $\pm\lambda_j^R \pm i\lambda_j^I$. The roots $\pm\lambda_j^{2s}$ are the corresponding ones obtained by solving (3.2).

j	λ_j^R	λ_j^I	λ_j^{2S}
1	0.790 764 41	0.525 126 70	0.769 265 10
2	0.770 641 99	1.584 745 20	0.384 626 65
3	0.387 753 87	0.506 240 73	0.201 040 90
4	0.386 237 01	1.522 850 69	0.063 339 91
5	0.202 074 33	0.503 939 40	
6	0.201 633 96	1.514 538 90	
7	0.063 610 60	0.503 283 39	
8	0.063 499 77	1.512 146 86	

Table 2. Complex roots $\lambda_j = \lambda_j^R + i\lambda_j^I$ ($j = 1$–23) of the system (2.6) corresponding to the state with lowest energy in the sector $r = 1$ of the spin $S = \frac{3}{2}$ model in a $L = 16$ site chain. There is a root at the origin ($\lambda_8 = 0$) and the other roots not shown in the table are obtained from these by taking the negative (complex conjugate) of the real (complex) roots. The five roots $0, \pm\lambda_7^R \pm i\lambda_7^I$ give us the structure that is tending toward ($L \to \infty$) a 3-string and a defect $\lambda^{\pm} = (0, \pm i)$. The roots $\pm\lambda_j^{2s}$ and 0 are the corresponding solution of (3.2).

j	λ_j^R	λ_j^I	λ_j^{2S}
1	0.594 225 42	0.930 295 54	0.574 727 80
2	0.528 275 24	0	0.302 851 83
3	0.343 764 11	0.954 389 34	0.136 881 41
4	0.285 664 06	0	0
5	0.186 056 05	0.964 783 93	
6	0.130 235 32	0	
7	0.059 296 25	0.968 820 10	

Table 3. The same as table 2 for the $L=20$ site chain. The five roots 0, $\pm\lambda_8^R \pm i\lambda_8^I$ are those tending toward a 3-string and a defect $\lambda^{\pm}=(0, \pm i)$.

j	λ_j^R	λ_j^I	λ_j^{2S}
1	0.667 056 87	0.933 472 57	0.651 553 01
2	0.601 955 63	0	0.383 961 60
3	0.418 407 45	0.958 863 55	0.227 289 19
4	0.363 864 19	0	0.107 066 40
5	0.266 046 74	0.970 074 61	
6	0.217 424 67	0	
7	0.149 512 96	0.975 284 32	
8	0.102 842 26	0	
9	0.048 364 74	0.977 408 48	

and $L=20$, given in tables 2 and 3 respectively, we clearly see that as L increases, instead, the two pure imaginary roots cluster in a 2-string $(-i/2, i/2)$, according to the string hypothesis, they prefer to form a defect $(-i, +i)$. The same fact occurs in sectors $r=4$ of spin $\frac{3}{2}$ and $r=1$ and 5 of spin 2. This implies that the string hypothesis, for some excited states, is not exact even in the infinite L limit. If we modify the string hypothesis (3.1) by introducing the above defects we obtain a different set of equations than (3.2) but the infinite L solution gives us the same dispersion relation as in (2.9) (Alcaraz and Martins 1988b). Calculating analytically the finite-size corrections as in the appendix, with these defects included, we found that while the first-order corrections remain the same as in the string hypothesis (see (3.5)) the logarithmic corrections are very different (Alcaraz and Martins 1988b). In these cases we use as the initial guess for the solution of equations (2.6) the solution obtained from these modified string-defect equations.

5. Results

In this section we present our main numerical results for the conformal anomaly and critical exponents for the models (2.3)-(2.5).

5.1. *Conformal anomaly*

The conformal anomaly can be extracted from the large L limit of the sequence

$$C_L = -(E_0 - e_\infty L)12L/\pi^2 \tag{5.1}$$

obtained from the relation (1.8) with the value $\zeta=\pi/2$ inferred from the momentum dispersion relation of the model (2.9) (von Gehlen *et al* 1986). In (5.1) E_0 is the ground state of the finite system and e_∞ is the bulk limit of the ground-state energy per particle, given by (2.8).

In table 4(a)-(c) we show, for spins 1 (L up to 84), $\frac{3}{2}$ (L up to 80) and 2 (L up to 64), these sequences for the true energies E_0 and for the energies E_0^{st} obtained by solving equations (3.2), derived under the string hypothesis. We also show, in these tables, the extrapolated results together with the conjectured ones. These extrapolations

Table 4. Finite-size sequences for the extrapolation of the conformal anomaly for the (*a*) spin-1, (*b*) spin-$\frac{3}{2}$ and (*c*) spin-2 model. E_0^{st} and E_0 are the ground-state energies obtained using and not using the string assumption (3.1), respectively. The conjectured results are given by (1.5).

L	$-E_0/L$	$-E_0^{st}/L$	$-(E_0-e_\infty L)12L/\pi^2$	$-(E_0^{st}-e_\infty L)12L/\pi^2$
(*a*)				
8	1.020 086	1.013 350	1.562 956	1.038 861
20	1.003 122	1.002 078	1.518 365	1.010 403
36	1.000 958	1.000 638	1.510 172	1.005 483
52	1.000 459	1.000 305	1.507 532	1.003 968
68	1.000 268	1.000 178	1.506 217	1.003 236
84	1.000 175	1.000 117	1.505 418	1.002 799
Extr	—	—	1.500 (4)	1.000 (2)
Conj	1.0	1.0	1.5	1.0
(*b*)				
8	1.217 487	1.206 658	1.894 045	1.045 097
20	1.196 908	1.195 230	1.829 403	1.012 984
32	1.194 607	1.193 957	1.818 656	1.007 835
44	1.193 917	1.193 574	1.814 368	1.005 839
56	1.193 622	1.193 411	1.812 046	1.004 784
68	1.193 469	1.193 326	1.810 575	1.004 130
80	1.193 379	1.193 276	1.809 549	1.003 682
Extr	—	—	1.800 (8)	1.000 (3)
Conj	1.193 147...	1.193 147...	1.8	1.0
(*c*)				
8	1.360 614	1.346 815	2.122 890	1.049 057
20	1.327 529	1.335 421	2.040 362	1.015 201
32	1.334 961	1.334 144	2.026 100	1.009 466
44	1.334 192	1.333 761	2.020 291	1.007 161
56	1.333 862	1.333 597	2.017 106	1.005 911
64	1.333 738	1.333 535	2.105 664	1.005 350
Extr	—	—	2.00 (1)	1.000 (5)
Conj	1.333...	1.333...	2.0	1.0

were done using the VBS approximants (Van den Broeck and Schwartz 1979, Hamer and Barber 1981). We see from these results that our numerical results are in excellent agreement with conjecture (1.5). The good convergence rate of these data indicate that, as in the case of the string energies (see (3.5*a*)), the correction term for (5.1) is of $o(l/\ln L)^3)$. If we extrapolate the difference between the true energies obtained by solving (2.6) and the string energies obtained from (2.11) we hope to cancel, if not totally at least partially, this next correction term. In fact, using these differences for the extrapolated values together with the exact result ($c=1$) for the string energies gives us slightly better estimates: $c=1.5001\pm0.0002$ ($S=1$), $c=1.8004\pm0.000\,(5)$ ($S=\frac{3}{2}$) and $c=2.0005\pm0.0005$ ($S=2$).

5.2. Scaling dimensions of the operators

The scaling dimensions of the operators governing the several correlation functions of the critical model can be estimated by using relations (1.7). To each r sector ($r=1,2,\ldots$) we expect that the mass gap amplitude of the lowest state E_r of this sector is related to a scaling dimension X_r of a primary operator. These dimensions can be estimated by the large L limit of the sequences

$$X_r(L)=(E_r-E_0)L^2/\pi^2. \tag{5.2}$$

In table 5 we present these sequences for the r sectors ($r=1$–5) of the $S=1,\frac{3}{2}$ and 2 models. Due to numerical instabilities we only calculate the lowest energy with $r=4$ sector of the spin-2 model for L up to 32. The extrapolated results of table 5 were

Table 5. Mass gap amplitudes and extrapolations for sectors $r=1$–5 of the (a) spin-1, (b) spin-$\frac{3}{2}$ and (c) spin-2 model (see (5.2)). The conjectured values are given by (1.6) and (6.1).

L	$X_1(L)$	$X_2(L)$	$X_3(L)$	$X_4(L)$	$X_5(L)$
(a)					
8	0.336 784	0.740 494	1.672 538	2.407 210	3.644 041
20	0.337 846	0.803 857	1.886 674	2.960 138	4.662 006
36	0.339 962	0.829 474	1.958 793	3.155 378	5.002 364
52	0.341 389	0.841 900	1.991 375	3.239 970	5.144 823
68	0.342 426	0.849 709	2.011 296	3.289 661	5.227 074
84	0.343 229	0.855 258	2.025 270	3.323 434	5.282 452
Extr	0.3 (5)	0.9 (4)	2.2 (5)	3.7 (7)	6.6 (1)
Conj	0.375	1.0	2.375	4.0	6.375
(b)					
8	0.284 978	0.624 958	0.991 413	1.851 179	2.500 350
20	0.277 241	0.654 902	1.114 668	2.112 626	3.068 554
32	0.276 194	0.665 726	1.155 600	2.186 552	3.234 594
44	0.275 981	0.671 960	1.177 753	2.224 442	3.317 209
56	0.275 994	0.676 234	1.192 276	2.248 737	3.368 604
64	0.276 084	0.678 461	1.199 627	2.260 931	3.393 840
80	0.276 203	0.681 991	1.210 970		3.431 857
Extr	0.2 (8)	0.7 (6)	1.3 (9)	2.6 (5)	4.0 (1)
Conj	0.3	0.8	1.5	2.8	4.3
(c)					
8	0.250 280	0.546 825	0.871 821	1.193 742	2.020 109
16	0.239 443	0.555 047	0.933 552	1.347 115	2.283 265
24	0.236 318	0.559 328	0.958 926	1.412 260	2.380 163
32	0.234 864	0.562 143	0.973 603	1.449 876	2.432 493
44	0.233 752	0.565 095	0.987 490		2.479 424
56	0.233 168	0.567 241	0.996 720		2.509 360
64	0.232 920	0.568 400	1.001 444		2.524 331
Extr	0.2 (3)	0.6 (3)	1.1 (6)	1.6 (5)	3.0 (4)
Conj	0.25	0.666...	1.25	2.0	3.25

obtained by using the VBS approximants (Van den Broeck and Schwartz 1979, Hamer and Barber 1981) and the conjectured values are those given by (1.6) and our subsequent analysis (see 6.1). We clearly see that the extrapolated results do not agree with the conjectured ones, which is not a surprise since from our analytical results of § 3 we already expect a correction term of order $(1/\ln L)$ in the estimator (5.2), producing a very slow convergence rate of that sequence. However the same type of correction occurs when we use in (5.3) the energies obtained by solving (3.2). We expect that a cancellation, if not complete at least partial, of this term may occur if we use, instead of (5.2), the sequence

$$D_r(L) = [(E_r - E_0) - (E_r^{\text{st}} - E_0^{\text{st}})]L^2/\pi^2 \tag{5.3}$$

where as before E_r^{st} $(r = 0, 1, 2, \ldots)$ denote the lowest energy in the sector r obtained by using the string hypothesis. Consequently the analytical result (3.5*b*) with the above sequence should give us a better estimate

$$X_r = D_r(\infty) + r^2/4S \tag{5.4}$$

for the scale dimensions.

In table 6(*a*)–(*c*) we show the sequences (5.3) for $r = 1$–5 and $S = 1$, $\frac{3}{2}$ and 2, respectively. We also show the VBS extrapolations as well as the estimates (5.4) in the lines marked VBS. With some exceptions for $S > 1$ (the cases denoted by the symbols † or ‡), which we shall discuss below, there is good agreement with the conjectured values (1.6) (see also (6.1)). Moreover the extrapolations in these cases are quite stable.

As we discussed in § 3 for certain values of the spin S and sector r (as for cases marked by the symbol † in table 6(*b*) and 6(*c*)), the string solution is not valid even in the $L \to \infty$ limit. Instead of having a simple sea of strings it appears in these cases that defects are not taken into account in the derivation of (3.2). Therefore the sequence (5.3) may still produce poor estimates because the first logarithmic correction of $E_r(L)$ and $E_r^{\text{st}}(L)$ may be very different. In these cases by introducing strings and defects we derived a different set of equations than (3.2). Solving these equations numerically we obtain the energies $E_r^{\text{def}}(L)$ having in principle a $o(1/L^2 \ln L)$ term which is a better approximation to the corresponding term of $E_r(L)$ than that obtained from $E_r^{\text{st}}(L)$. In these cases the sequence

$$F_r^S(L) = [(E_r - E_0) - (E_r^{\text{def}} - E_0^{\text{st}})]L^2/\pi^2 \tag{5.5}$$

may produce better estimates for the scaling dimensions, where we have used the fact that in the $L \to \infty$ limit the ground state does not show defects, being exactly represented by a sea of $2S$-strings. It is important to mention (Alcaraz and Martins 1988b) that the $o(1/L^2)$ correction term for the energies $E_0^{\text{def}}(L)$ are the same as the corresponding $E_0^{\text{st}}(L)$ and consequently the estimate for these dimensions are

$$X_r = F_r^S(\infty) + r^2/4S. \tag{5.6}$$

In table 7 we show the sequences (5.5) for the cases where these defects occur (denoted by † in table 6(*b*) and (*c*)). We then see clearly that the extrapolated results and the estimate (5.6), for the line marked VBS, are in much better agreement with the conjectured results. In the extrapolation procedure we introduce a small parameter ε to measure the stability of the converged values (Hamer and Barber 1981). In all these cases these values are very stable even for reasonably large values of ε which imply that the string-defect solution has a $o(1/L^2 \ln L)$ correction term not much different, if not equal, to the corresponding term of the true energy E_r obtained by solving (2.6).

Table 6. Finite-size sequences of the quantities $D_r(L)$, $r = 1$–5 for the (a) spin-1, (b) spin-$\frac{3}{2}$ and (c) spin-2 model (see (5.3)). The extrapolated results denoted by VBS are obtained from (5.4) and the conjectured values from (1.6) and (6.1).

L	$D_1(L)$	$D_2(L)$	$D_3(L)$	$D_4(L)$	$D_5(L)$
(a)					
8	0.118 550	0.014 8478	0.061 6628	0.036 8408	0.045 0017
20	0.121 419	0.005 3599	0.088 5157	0.018 4639	0.064 9935
36	0.122 445	0.003 0060	0.101 4681	0.009 9641	0.077 3298
52	0.122 889	0.002 0877	0.107 3706	0.006 6169	0.087 2656
68	0.123 146	0.001 8149	0.110 7617	0.004 9272	0.093 6852
84	0.123 317	0.001 5791	0.112 9719	0.003 9367	0.098 1576
Extr	0.124 (6)	0.000 (3)	0.124 (4)	0.000 (7)	0.124 (5)
VBS	0.374 (6)	1.000 (3)	2.374 (4)	4.000 (7)	6.374 (5)
Conj	0.375	1.0	2.375	4.0	6.375
(b)					
8	0.125 741†	0.078 182	0.039 856	0.050 982†	0.065 433‡
20	0.127 450†	0.093 447	0.019 095	0.060 216†	0.055 793‡
32	0.128 449†	0.101 362	0.012 824	0.070 177†	0.058 342‡
44	0.129 062†	0.106 101	0.009 826	0.077 322†	0.062 747‡
56	0.129 486†	0.109 307	0.008 074	0.082 623†	0.067 051‡
64	0.129 704†	0.110 941	0.007 262	0.085 466†	0.069 668‡
80	0.130 046†	0.113 453	0.006 114		0.074 263‡
Extr	0.133 (1)	0.133 (2)	0.000 (9)	0.0 (9)	0.0 (9)
VBS	0.299 (7)	0.799 (8)	1.500 (9)	2.7 (5)	4.2 (6)
Conj	0.3	0.8	1.5	2.8	4.3
(c)					
8	0.117 443†	0.074 959	0.066 114‡	0.065 623	0.063 100†
16	0.117 251†	0.090 187	0.062 163‡	0.045 327	0.041 345†
24	0.117 858†	0.100 351	0.064 649‡	0.035 572	0.041 100†
32	0.118 379†	0.107 251	0.067 659‡	0.029 678	0.043 087†
44	0.118 976†	0.114 341	0.071 698‡		0.046 773†
56	0.119 423†	0.119 270	0.075 026‡		0.050 284†
64	0.119 665†	0.121 831	0.076 915‡		0.052 412†
Extr	0.125 (1)	0.166 (6)	0.0 (9)	0.00 (1)	0.0 (8)
VBS	0.500 (1)	0.666 (6)	1.13 (4)	2.00 (1)	3.2 (1)
Conj	0.5	0.666...	1.25	2.0	3.25

The cases denoted by ‡ in table 6(b) and (c), specifically sector 5 of spin $\frac{3}{2}$ and sector 3 of spin 2, also do not give us good estimates for the scaling dimensions. These states do correspond, in the $L\to\infty$ limit, to a sea of strings and the poor convergence rate in these cases are of a different nature than the preceding cases denoted by †. In these cases the string solution itself, although valid in $L\to\infty$, does not give a good value for the $o(1/L^2 \ln L)$ term. We conclude from this that the imaginary part of the

Table 7. Finite-size sequences of the quantities $F_r^S(L)$ (see (5.5)). The first (last) two columns refer to the spin-$\frac{3}{2}$ (2) model. The extrapolated results denoted by VBS are obtained from (5.6) and the conjectured values from (1.6) and (6.1).

L	$F_1^{3/2}(L)$	$F_4^{3/2}(L)$	$F_1^2(L)$	$F_5^2(L)$
8	0.340 819	0.459 312	0.338 704	0.510 718
20	0.227 749	0.373 160	0.221 006	0.423 842
32	0.195 369	0.319 887	0.188 362	0.363 756
44	0.179 884	0.287 819	0.172 872	0.326 357
56	0.170 758	0.266 372	0.163 769	0.300 871
64	0.166 494	0.255 575	0.159 519	0.287 896
80	0.160 426			
Extr	0.1333 (2)	0.133 (0)	0.12 (5)	0.12 (6)
VBS	0.2999 (8)	2.799 (6)	0.50 (0)	3.25 (1)
Conj	0.3	2.8	0.5	3.25

roots of (2.6) being not fixed (integer or half-integer), as in (3.2), produce different contributions not only of order $1/L^2$ but also of order $(1/L^2 \ln L)$. In order to obtain better estimates, in these cases, we extrapolate the difference of these energies with other states, testing the stability of the convergence. A good convergence for sector 5 of spin $\frac{3}{2}$ and sector 3 of spin 2 was obtained by using the sequences $S_1 = [(E_5 - E_5^{\mathrm{st}}) - (E_4 - E_4^{\mathrm{st}})]L^2/\pi^2$ and $S_2 = [(E_5 - E_5^{\mathrm{def}}) - (E_3 - E_3^{\mathrm{st}})]L^2/\pi^2$, respectively. These sequences give the extrapolated results $S_1(\infty) = -0.00\,(1)$ and $S_2(\infty) = 0.00\,(2)$ producing, from (3.5b) and the results of tables 4 and 5, the estimates $X_5 = 4.29\,(0)$ (spin $\frac{3}{2}$) and $X_3 = 1.25\,(2)$ (spin 2) to be compared with our conjecture (see 6.1): $X_5 = \frac{103}{24} = 4.291\,66\ldots$ (spin $\frac{3}{2}$) and $X_3 = \frac{5}{4} = 1.25$ (spin 2).

6. Conclusion and summary

In this paper we calculate the conformal anomaly and scaling dimensions of the operators governing the criticality of the spin $S = 1, \frac{3}{2}$ and 2 Takhtajan–Babujian models (2.3)–(2.5). These quantities were calculated by exploiting their relations with the eigenspectrum of the model in a finite chain (1.7) and (1.8). In conjunction with analytical (§ 3 and the appendix) and numerical results (§§ 4 and 5) we analyse the Bethe ansatz equations for these models for finite chain and periodic boundary conditions. This conjunction was necessary because the presence of logarithmic finite-size corrections produce poor convergence rates. These logarithmic corrections, as in the Heisenberg model ($S = 1$), we believe indicate that the operator responsible for the corrections to scaling is marginal ($X = 2$).

In fact, by introducing an anisotropy constant in the Takhtajan–Babujian model, still preserving integrability (Sogo *et al* 1983, Babujian and Tsvelick 1986, Kirillov and Reshetikhim 1987a, b) we can show (Alcaraz and Martins 1988b) that the operator responsible for the corrections to scaling is irrelevant ($X > 2$) for general values of the anisotropy except at the isotropic point (the Takhtajan–Babujian model) where it becomes marginal, originating the logarithmic corrections.

Our numerical results are in good agreement with the conjectured values (1.5) and (1.6) (Affleck 1986a, b, c, Affleck and Haldane 1987). This agreement strongly supports the conjecture that the Wess-Zumino-Witten-Novikov non-linear σ model, with topological charge $k=2$ and group $\mathrm{G}=\mathrm{SU}(2)$, is the underlying field theory describing the criticality of these spin models.

Our numerical results also indicate scaling dimensions of irrelevant operators not given in the conjecture (1.6). From the string assumption (3.1) the lowest eigenenergy of a given sector r corresponds to a configuration of complex roots having a single $\{r/2S\}$-string in a sea of $2S$-strings, where $\{m/n\}$ is the rest of the ratio m/n. Our numerical results indicate that the corresponding amplitude, for these sectors, has a contribution which depends on $\{r/2S\}$ beyond the simple string type contribution: $r^2/4S$. All our results lead us to the following conjecture for the scaling dimensions associated with the lowest energy of the sector r of the spin-S model:

$$X_r=\frac{r^2-R^2}{4S}+\frac{(R+2)R}{4(1+S)} \qquad r=1,2,\ldots \tag{6.1}$$

where R is the rest of the ratio $r/2S$. The conjecture (6.1) reproduces (1.6) for $r\leqslant 2S$ and agrees with our results of § 5 for $r>2S$.

Since the above dimensions are obtained from the spin models with periodic boundary conditions we expect to obtain them from the modular invariant solution of the SU(2) Kac-Moody model with central charge k (Gepner and Witten 1986, Capelli *et al* 1987). In the case where k is odd a single solution (A connected with the simply laced Lie algebra A_{k+1}) occurs while, when k is even, other solutions occur (A, D and E connected with simply laced Lie algebra A, D and E). The above dimensions (6.1) and our results (Alcaraz and Martins 1988b) of the anisotropic version of the Takhtajan-Babujian models indicate that they are realisations of the modular invariant solution of type A.

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Appendix. The leading finite-size corrections

In this appendix we will derive, based on the string assumption (3.1), the leading finite-size corrections for the eigenenergies of the spin-S Takhtajan-Babujian models. Our calculation will be based on a method developed by Woynarovich and Eckle (1987a, b) (see also Hamer *et al* 1987).

We will calculate the finite-size corrections of the lowest energies E_L^r of the sector $r=0,1,2,\ldots$, of these spin models. These energies are given by (3.2) and (3.4) with Q_j^n given by (3.3). For a given distribution of strings $\{\nu_k\}$ it is convenient to introduce the density of roots $\sigma_L^r(\lambda)$ for the $2S$-strings (λ_j^{2s}, $j=1,2,\ldots,\nu_s$) in the sector r of the finite system by

$$\sigma_L^r(\lambda)=\mathrm{d}Z_L^r/\mathrm{d}\lambda \tag{A1a}$$

where

$$Z_L^r(\lambda_j^{2S}) = \frac{Q_j^r}{2\pi} = \frac{1}{2\pi}\left(\psi_{2S,2S}(\lambda_j) - \frac{1}{L}\sum_{k=1}^{\infty}\sum_{i=1}^{\nu_k} \Xi_{2S,k}(\lambda_j^{2S} - \lambda_i^k)\right). \tag{A1b}$$

The above functions ψ and Ξ are defined in (3.2). When $L \to \infty$ the roots tend toward a continuous distribution with density given by

$$\sigma_\infty(\lambda) = \frac{1}{2\pi}\psi'_{2S,2S}(\lambda) - \sigma_\infty(u)\Xi'_{2S,2S}(\lambda - u)\,\mathrm{d}u \tag{A2}$$

where the prime indicates the derivative. The above integral equation has the solution (Takhtajan 1982, Babujian 1982)

$$\sigma_\infty(\lambda) = \frac{1}{2\cosh(\pi\lambda)} \tag{A3}$$

and from (3.4) the energy per site is given by

$$e_\infty = -\tfrac{1}{2}\int_{-\infty}^{+\infty} \sigma_\infty(\lambda)\psi'_{2S,2S}(\lambda)\,\mathrm{d}\lambda. \tag{A4}$$

Using equations (3.2), (3.4) and (A1)-(A4) and after some lengthy manipulations we can express the difference of the energy and density of roots from their bulk limit ($L \to \infty$) values by

$$E^r/L - e_\infty = -\pi\int_{-\infty}^{+\infty} \sigma_\infty(u)S(u)\,\mathrm{d}u \tag{A5}$$

and

$$\sigma_L^r - \sigma_\infty = -\frac{1}{L}\Xi'_{2S,2S-r} - \frac{1}{2\pi}\int_{-\infty}^{+\infty} P(\lambda - u)S(u)\,\mathrm{d}u \tag{A6}$$

respectively, where

$$S(u) = \frac{1}{L}\sum_j \delta(\lambda_j - u) - \sigma_L^r(u) \tag{A7}$$

$$P(\lambda) = \int_{-\infty}^{+\infty} \exp(\mathrm{i}\lambda w)K(w)\,\mathrm{d}w \tag{A8}$$

$$[1 - K(w)]^{-1} = G_+(w)G_-(w) \tag{A9}$$

and

$$G_\pm(w) = \frac{(\pi/S)^{1/2}\Gamma(\pm\mathrm{i}w/2)[\exp(\pm\mathrm{i}\pi/2)Sw/e\pi]^{\pm\mathrm{i}|w|S/2}}{\Gamma(\mp\mathrm{i}sw/\pi)\Gamma(\frac{1}{2}\mp\mathrm{i}w/2\pi)}. \tag{A10}$$

According to the method of Woynarovich and Eckle (1987a, b) (see also Hamer *et al* 1987), for large L, the leading finite-size corrections are given by

$$E^r/L - e_\infty = 2\pi\int_\Lambda^\infty \sigma_\infty(\lambda)\sigma_L^r(\lambda)\,\mathrm{d}\lambda - \frac{1}{2L}\sigma_\infty(\Lambda) - \frac{\sigma_\infty(\Lambda)}{12L^2\sigma_L^r(\Lambda)} \tag{A11}$$

$$\begin{aligned}\sigma_L^r(\lambda) - \sigma_\infty(\lambda) &= \frac{1}{2\pi}\int_\Lambda^\infty \sigma_L^r(u)p(\lambda - u)\,\mathrm{d}u - \frac{p(\lambda-\Lambda)}{4\pi L} + \frac{1}{12L^2}\frac{p'(\lambda-\Lambda)}{\sigma_L^r(\Lambda)}\\ &\quad + \left(\int_{-\infty}^{-\Lambda} \sigma_L^r(u)\frac{P(\lambda - u)}{2\pi}\,\mathrm{d}u - \frac{P(\lambda+\Lambda)}{2\pi L} - \frac{P'(\lambda+\Lambda)}{24L^2\sigma_L^r(\Lambda)} - \frac{\Xi'}{L}2S, 2S-r\right)\end{aligned} \tag{A12}$$

where Λ is the largest magnitude root determined by the boundary condition

$$\int_{\Lambda}^{\infty} \sigma_L^r(\lambda)\,\mathrm{d}\lambda = \frac{(1-2r)}{2L}. \tag{A13}$$

The first-order corrections, which are $o(1/L^2)$, can be calculated by dropping the large bracket terms in (A12), which are responsible for logarithmic corrections. Defining

$$R(\lambda) = P(\lambda)/2\pi \qquad f(\lambda) = \sigma_\infty(\lambda+\Lambda) \tag{A14}$$

$$X^r(\lambda) = \sigma_L^r(\lambda+\Lambda) \qquad t = \lambda - \Lambda \tag{A15}$$

equation (A12) now becomes

$$X^r(t) = f(t) + \int_0^\infty X^r(u)R(t-u)\,\mathrm{d}u - \frac{R(t)}{L} + \frac{R'(t)}{12L^2\sigma_L^r(\Lambda)} \tag{A16}$$

which is the standard form of the Wiener-Hopf equation (see, for example, Morse and Feshbach 1953). To solve this equation we introduce the Fourier transforms

$$\tilde{X}_\pm^r(w) = \int_{-\infty}^{+\infty} \exp(\mathrm{i}wt)X_\pm^r(t)\,\mathrm{d}t \qquad X_\pm^r(t) = \begin{cases} X^r(t) & t \gtrless 0 \\ 0 & t \lessgtr 0 \end{cases} \tag{A17}$$

and the corresponding Fourier pairs $f(t) \leftrightarrow \tilde{f}(w)$, $R(t) \leftrightarrow \tilde{R}(w)$. After some algebra we find that $\tilde{X}_+^r(w)$ is given by

$$\tilde{X}_+^r(w) = C^r(w) + G_+(w)[Q_+(w) + P(w)] \tag{A18}$$

where

$$C^r(w) = \frac{1}{2L} - \frac{\mathrm{i}w}{12L^2\sigma_L^r(\Lambda)} \qquad Q_+(w) = \frac{G_-(\mathrm{i}\pi)\exp(-\pi\Lambda)}{\pi - \mathrm{i}w} \tag{A19}$$

$$P(w) = -\frac{1}{2L} + \frac{\mathrm{i}g}{12L^2\sigma_L^r(\Lambda)} - \frac{\mathrm{i}w}{12L^2\sigma_L^r(\Lambda)} \qquad g = \mathrm{i}\pi(\tfrac{1}{4} - 1/12S). \tag{A20}$$

From equation (A16) and definition (A15) we obtain

$$\frac{G_-(\mathrm{i}\pi)\exp(-\pi\Lambda)}{\pi} = \frac{1}{2L} + \frac{r}{LG_+(0)} - \frac{\mathrm{i}g}{12L^2\sigma_L^r(\Lambda)} \tag{A21}$$

and

$$\sigma_L^r(\Lambda) = \frac{g^2}{24L^2\sigma_L^r(\Lambda)} + \frac{\mathrm{i}g}{2L} + G_-(-\mathrm{i}\pi)\exp(-\pi\Lambda). \tag{A22}$$

Finally, using (A18), (A21) and (A22) in (A11) and approximating $\sigma_\infty(\Lambda) \simeq \exp(-\pi\Lambda)$ we obtain the first-order correction for the lowest energy of sector r:

$$E^r/L - e_\infty = -\frac{\pi^2}{L^2}\left(\frac{1}{12} - \frac{r^2}{4S}\right) \qquad r = 0, 1, 2, \ldots. \tag{A23}$$

If we now use the above solution (A21) and (A22) and include the large brackets of (A12) we can solve perturbatively the Wiener-Hopf equation (A16) in order to obtain the next correction terms (Alcaraz and Martins 1988b):

$$E^0/L - e_\infty = -\frac{\pi^2}{12L^2} + o\left(\frac{1}{L^2(\ln L)^3}\right) + o\left(\frac{\ln(\ln L)}{L^2(\ln L)^4}\right) \tag{A24}$$

for the ground-state energy ($r=0$) and

$$\frac{E^r}{L}-\frac{E^0}{L}=\frac{\pi^2 r^2}{4SL^2}+o\left(\frac{1}{L^2 \ln L}\right)+o\left(\frac{n(\ln L)}{L^2(\ln L)^2}\right) \qquad \text{(A25)}$$

for the mass gap amplitude of sector r.

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